Assignment 5

Hand in no. 1, 4, 8 and 9 by October 11, 2018.

1. Show that whenever d is a metric defined on X, then
$$\rho(x,y) \equiv \frac{d(x,y)}{1+d(x,y)}$$
is also a metric on X. A sequence converges in d if and only if it converges in ρ.

2. Show that d_2 is stronger than d_1 on $C[a,b]$ but they are not equivalent. Hint: Construct a sequence $\{f_n\}$ in $C[0,1]$ satisfying $\|f_n\|_1 \to 0$ but $\|f_n\|_2 \to \infty$ as $n \to \infty$.

3. Consider the functional Φ defined on $C[a,b]$
$$\Phi(f) = \int_a^b \sqrt{1+f^2(x)}\ dx.$$
Show that it is continuous in $C[a,b]$ under both the supnorm and the L^1-norm. A real-valued function defined on a space of functions is traditionally called a functional.

4. Consider the functional Ψ defined on $C[a,b]$ given by $\Psi(f) = f(x_0)$ where $x_0 \in [a,b]$ is fixed. Show that it is continuous in the supnorm but not in the L^1-norm. Suggestion: Produce a sequence $\{f_n\}$ with $\|f_n\|_1 \to 0$ but $f_n(x_0) = 1,\ \forall n$. Ψ is called an evaluation map.

5. Let Φ be a continuously differentiable function on $\mathbb{R}$. Define a function from $C[0,1]$ to itself by $G(f)(x) = \Phi(f(x))$. Show that G is continuous.

6. Let K be a continuous function defined on $[0,1] \times [0,1]$ and consider the map
$$T(f)(x) = \int_0^1 K(x,y) f(y) dy\ .$$
Show that this map maps $(C[0,1], \|\cdot\|_1)$ to $(C[0,1], \|\cdot\|_\infty)$ continuously.

7. Let A and B be two sets in (X,d) satisfying $d(A,B) > 0$ where
$$d(A,B) \equiv \inf\{d(x,y) :\ (x,y) \in A \times B\}\ .$$
Show that there exists a continuous function f from X to $[0,1]$ such that $f \equiv 0$ in A and $f \equiv 1$ in B. This problem shows that there are many continuous functions in a metric space.

8. In class we showed that the set $P = \{f :\ f(x) > 0, \forall x \in [a,b]\}$ is an open set in $C[a,b]$. Show that it is no longer true if the norm is replaced by the L^1-norm. In other words, for each $f \in P$ and each $\varepsilon > 0$, there is some continuous g which is negative somewhere such that $\|g - f\|_1 < \varepsilon$.

9. Show that $[a,b]$ can be expressed as the intersection of countable open intervals. It shows in particular that countable intersection of open sets may not be open.

10. Optional. Show that every open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals. Suggestion: Introduce an equivalence relation $x \sim y$ if x and y belongs to the same open interval in the open set and observe that there are at most countable many such intervals.

11. Fill in a proof of Proposition 2.8 (b).